Assignment 5 Supplement
Algorithm Design and Analysis

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3 Unique Cut

Here is the solution excerpted from stanford[1].

First compute a minimum $s-t$ cut C[2], and define its volume by $|C|$. Let $e_1, e_2, ..., e_k$ be the edges in C. For each e_i, try increasing the capacity of e_i by 1 and compute a minimum cut in the new graph. Let the new minimum cut be C_i, and denote its volume(in the new graph) as $|C_i|$. If $|C| = |C_i|$ for some i, then clearly C_i is also a minimum cut in the original graph and $C \neq C_i$, so the minimum cut is not unique. Conversely, if there is a different minimum cut C' in the original graph, there will be some $e_i \in C$ that is not in C', so increasing the capacity of that edge will not change the volume of C', thus $|C| = |C_i|$. In conclusion, the graph has a unique minimum cut if and only if $|C| < |C_i|$ for all i. The algorithm takes at most $m + 1$ computing of minimum cuts, and therefore runs in polynomial time.

助教解法：

给定图 $G = (V, E)$，在图 G 上跑最大流算法，得到残余图 G_f。在 G_f 中，如果和 s 以及 t 相连的点的集合 $V' \neq V$，则最小割不唯一，否则唯一。

假设有一个点 $v \in V - V'$，则和 v 相连的入边和出边都已经满流了（否则 v 肯定在残余图中有边），那么最小割既可以割 v 的入边，也可以割 v 的出边，所以割不唯一。

4 Problem Reduction

Suppose we have a matrix like Figure 1, for each point except s and t, we split it into two points, we get Figure 2.

We set the capacity of all edges as 1, which forces each point can't be walk through more than once. We set the cost between sub-points of one original point as the original number. One path $(s \to t \to s)$ without duplicate points equals to two paths $(s \to t), (s \to t)$ without duplicate points, so we set the objective of the extended network as 2.

By running minimum cost flow algorithm, we can find the minimal cost from s to t and back to s.

[1] http://stanford.edu/~rezab/discrete/Midterm/pmidtermIsoln.pdf

[2] http://stackoverflow.com/a/4490705/2468587

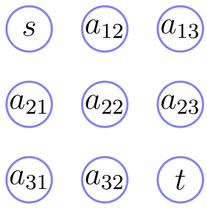

Figure 1: Original matrix with numbers.

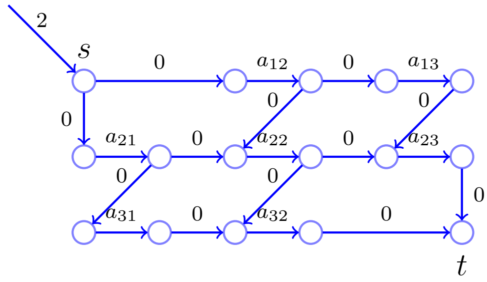

Figure 2: Extended network for Figure 1,$C(e) = 1$ for all edges and the weight shows on the line, the objective is 2.

6 Maximum Cohesiveness

Here is the solution excerpted from iitd[3].
Given a graph $G =< V, E, W >$, we construct a network graph G' as follows.

- There is a vertex v corresponding to every vertex v in G.
- There is a vertex v_{ij} corresponding to each edge e_{ij} in G.
- There is a source s and a sink vertex t.
- Vertex v_{ij} has edges to vertex i and j. This has capacity ∞.
- There is an edge from s to all vertexes v_{ij}. These edges have capacity w_{ij}.
- There is an edge from every vertex v to t with capacity α.

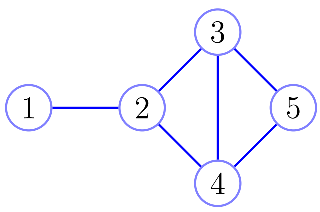

Figure 3: Original undirected graph G.

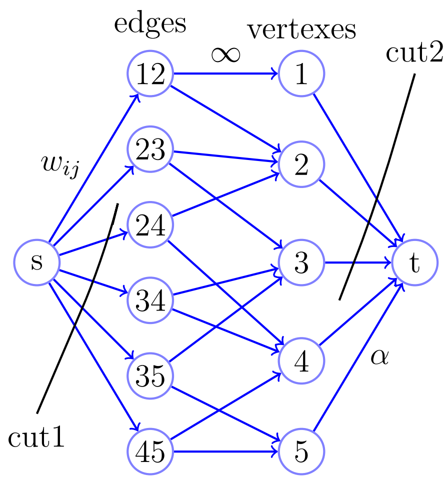

Figure 4: Extended network G'.

Let (A, B) be a min-cut in G'. Let S be the vertexes on the right that are in B.

- Claim 1: If v_{ij} is in A, then both i and j are in A, otherwise (v_{ij}, i) or (v_{ij}, j) can be augmented, it isn't a maximal flow.
- Claim 2: If i and j is in A, then v_{ij} is in A.

[3] http://www.cse.iitd.ac.in/~rjaiswal/2011/csl356/Notes/Week-11/lec-2.pdf

- Claim 3: The capacity of min cut, $C(A,B) = e(Total) - e(S) + |S|\alpha$
- Claim 4: There is a subset S with cohesiveness $\geq \alpha$ **if and only if** $C(A,B) \leq e(Total)$.

If Claim 4 holds, we have $C(A,B) \leq e(Total) \Leftrightarrow e(Total) - e(S) + |S|\alpha \leq e(Total)$ $\Leftrightarrow e(S)/|S| \geq \alpha \Leftrightarrow S$ is the maximum cohesiveness.

助教解法：
因为 $e(Total) = e(S) + e(\bar{S}) + e(S\bar{S})$, 如果 $C(A,B) = e(\bar{S}) + e(S\bar{S}) + |S|\alpha$, 那么有

$$\begin{aligned} C(A,B) &\leq e(Total) \\ \Rightarrow \quad e(\bar{S}) + e(S\bar{S}) + |S|\alpha &\leq e(S) + e(\bar{S}) + e(S\bar{S}) \\ \Rightarrow \quad \alpha &\leq e(S)/|S| \end{aligned}$$

所以 S 是 maximum cohesiveness。下面我们来证明 $C(A,B) = e(\bar{S}) + e(S\bar{S}) + |S|\alpha$。

假设最小割如 Figure 4 中的 cut1 和 cut2，则 $S = \{1,2,3\}$。注意割不一定要是连续的一条线，只要把 s 和 t 割开，使得 s 和 t 无法到达即可。

1. $e(S) \notin C(A,B)$. 因为 S 被割，对应残余图中边 $<S,t>$ 已经满流，所以左边不能再流过去了，即 $e(S) \notin C(A,B)$。比如 Figure 4 右边割了 $S = \{1,2,3\}$，则不能再割 S 内部的边，如 $<1,2>$ 和 $<2,3>$，即边 $<s,12>$ 和 $<s,23>$ 不会被割。
2. $e(S\bar{S}) \in C(A,B)$. 如果不割 $e(S\bar{S})$，则边还没满流，还可以增加流量，即没有达到最大流，和当前是最小割矛盾。比如 $3 \in S$ 且 $4 \notin S$，如果边 $<3,4>$ 即 $<s,34>$ 没有被割，则还可以通过 $s \to 34 \to 4 \to t$ 增加流量，不满足最大流，$C(A,B)$ 也就不是最小割了，矛盾。所以 $e(S\bar{S}) \in C(A,B)$。
3. $e(\bar{S}) \in C(A,B)$. 和 2 的证明类似，因为 $4 \notin S, 5 \notin S$，如果边 $<s,45>$ 不被割，则不是最大流，所以边 $<s,45>$ 被割，即 $e(\bar{S}) \in C(A,B)$。

由此得证 $C(A,B) = e(\bar{S}) + e(S\bar{S}) + |S|\alpha$，进而得到 $\alpha \leq e(S)/|S|$，S 为 maximum cohesiveness。